

Sudoku Symmetry Group

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Abstract

The mathematical aspects of the popular logic game Sudoku incorporate a significant number of the group theory concepts. In this note, we describe all symmetric transformations of the Sudoku grid. We do not intend to obtain a new strategy of solving Sudoku and do not describe basic ideas of the game which can be found in numerous other sources.

1 Symmetric transformations of the Sudoku grid

To define a Sudoku grid we will use the usual matrix notations, i.e:

1. $a_{i,j}$ - this is the grid cell with the row index i and the column index j .
2. The Sudoku 3×3 squares we will call blocks.
3. Three horizontal sectors (bands) are defined by the following sets of rows: $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$.
4. Three vertical sectors (stacks) are defined by the following sets of columns: $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$.

Let us fix some completely filled Sudoku grid. The natural question arises: How many new Sudoku grids can be obtained from the fixed one by different symmetric transformations? We will give a complete description of all symmetric transformations and will show that any symmetric transformation S of the Sudoku grid can be represented as some permutation of the rows in the horizontal sectors, permutation of the horizontal sectors itself and matrix transposition. We will prove that all other symmetric transformations that are described by many authors can be obtained by some combination of the main three mentioned above.

We should introduce some notations:

1. $A_1 = \{E\}$ where E - is the identical transformation.

2. $A_2 = \{d\}$ where d - is the matrix transposition.
3. $A_3 = \{r_i\}$ where index $i = \overline{1, 6^4 - 1}$ and $r_i \neq E$.

For any element from A_3 at least one of the following conditions holds true: $r_i^2 = E$, $r_i^3 = E$, $r_i^6 = E$, $r_i^9 = E$, $r_i^{12} = E$. Where r_i is the permutation of the rows in some horizontal sector, permutation of sectors itself or both. Each r_i is related to some substitution of the following type:

$$r \rightarrow \begin{pmatrix} 1\ 2\ 3 & 4\ 5\ 6 & 7\ 8\ 9 \\ \alpha_1\alpha_2\alpha_3 & \beta_1\beta_2\beta_3 & \gamma_1\gamma_2\gamma_3 \end{pmatrix}.$$

Where each triplet $(\alpha_1\alpha_2\alpha_3)$, $(\beta_1\beta_2\beta_3)$, $(\gamma_1\gamma_2\gamma_3)$ this is $(1, 2, 3)$ or $(4, 5, 6)$ or $(7, 8, 9)$ i.e these are triplet's permutations. The total number of them is 6. The total number of the permutations inside the triplets equals 6^3 . Therefore the total number of all possible r_i is equal to $(6^4 - 1)$ (we do not take into account the identical one). These permutations define a subgroup in S_9 (if we add the identical one of course).

4. $A_4 = \{r_k d\}$ where $r_k \neq E$ and $k = \overline{1, 6^4 - 1}$.
5. $A_5 = \{dr_l\}$ where $r_l \neq E$ and $l = \overline{1, 6^4 - 1}$.
6. $A_6 = \{dr_j d\}$ where $r_j \neq E$ and $j = \overline{1, 6^4 - 1}$.

Where A_6 is a permutation of columns in a vertical sector, or permutation of vertical sectors itself, or both at the same time.

7. $A_7 = \{r_s dr_t\}$ where $r_s \neq E$, $r_t \neq E$ and $s, t = \overline{1, 6^4 - 1}$.
8. $A_8 = \{r_\lambda dr_\mu d\}$ where $r_\lambda \neq E$, $r_\mu \neq E$ and $\lambda, \mu = \overline{1, 6^4 - 1}$.

Where A_8 is a set of permutations rows and columns inside sectors and permutations of sectors itself.

Desired group of symmetries $S = \bigcup_{i=1}^8 A_i$. To obtain a complete description of group S we will need the following relation.

Proposition 1.1.

$$r_\lambda(dr_\mu d) = (dr_\mu d)r_\lambda, \quad \forall \lambda, \forall \mu (*)$$

Proof.

$$\begin{aligned} a_{ij}(r_\lambda) &\rightarrow a_{mj}(d) \rightarrow a_{jm}(r_\mu) \rightarrow a_{nm}(d) \rightarrow a_{mn} \\ a_{ij}(d) &\rightarrow a_{ji}(r_\mu) \rightarrow a_{ni}(d) \rightarrow a_{in}(r_\lambda) \rightarrow a_{mn} \end{aligned}$$

□

The next set of relations easy to obtain from the 1.1.

Corollary 1.2.

- $A_i \cap A_j = \emptyset$ if $i \neq j$.
- $A_i \cdot A_j$ belongs to the union $\forall i, \forall j$ sets from $\{A_i\}$ where $i = \overline{1, 8}$.

(for example: $A_7 \cdot A_8 \subset A_7 \cup A_2 \cup A_4 \cup A_5$)

Proof.

$$a = r_s dr_t r_\lambda dr_\mu d = r_s (dr_t r_\lambda d) r_\mu d = (dr_t r_\lambda d) r_s r_\mu d = (dr^* d) r^{**} d = r^{**} dr^*$$

where $r^* = r_t r_\lambda$, $r^{**} = r_s r_\mu$.

$$\begin{cases} \text{if } r^* = E \text{ and } r^{**} = E & \text{then } a \in A_2. \\ \text{if } r^* = E \text{ and } r^{**} \neq E & \text{then } a \in A_4. \\ \text{if } r^* \neq E \text{ and } r^{**} = E & \text{then } a \in A_5. \\ \text{if } r^* \neq E \text{ and } r^{**} \neq E & \text{then } a \in A_7. \end{cases}$$

As a result

$$A_7 \cdot A_8 \subset A_2 \cup A_4 \cup A_5 \cup A_7$$

using the same approach one can show it for $\forall i, \forall j$ using the relation 1.1. □

It's easy to count that

$$\overline{\overline{S}} = \sum_{i=1}^8 \overline{\overline{A_i}};$$

where

$$\overline{\overline{S}} = 2 + (6^4 - 1) + (6^4 - 1) + (6^4 - 1) + (6^4 - 1) + (6^4 - 1)^2 + (6^4 - 1)^2 = 2 \cdot 6^8.$$

If the add to the obtained group S the group of interchange of the digits (we will call this new group $O = \{o_{ij}\}; i = \overline{1, 9!}$). The group O_i will be equivalent to the group S_9 (9th order substitution). It's obvious that O_i commutes with any element of the group S because O_i commutes with d and r_i .

$$\overline{\overline{O}} = 9!.$$

We obtain that $S \cdot O = O \cdot S$

$$\overline{\overline{S \cdot O}} = 2 \cdot 6^8 \cdot 9!$$

The same results can be found in some other articles (for example in the article of Royle Gordon). Let us finally show that the constructed group S exhausts all possible symmetries of the Sudoku grid.

Permutations of the rows and columns inside the sectors, permutations of horizontal and vertical sectors, transposition are exhausted by sets $\{A_2\}$, $\{A_3\}$ and $\{A_6\}$. All the rest symmetries can be obtained as a combination of some elements of the group S .

1. H - is a symmetry with respect to the 5th row.

$$a_{ij} \rightarrow a_{10-i,j} \quad \text{or} \quad H \rightleftharpoons r_1$$

where

$$r_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}; \quad r_1^2 = E$$

We can identify $H = r_1$, $r_1 \in A_3$. Elements of the substitution r_1 with the numbers of the grid rows.

2. H^1 is a symmetry with respect to the 5th column.

$$a_{ij} \rightarrow a_{i,10-j} \quad \text{or} \quad H^1 = dr_1d; \quad H^1 \in A_6.$$

3. Symmetry with respect to the additional diagonal

$$a_{ij} \rightarrow a_{10-i,10-j} \quad \text{or} \quad D = r_1dr_1; \quad D \in A_7.$$

4. V - rotation of rows to $\pi/2$ clockwise

$$a_{ij} \rightarrow a_{j,10-i} \quad \text{or} \quad V = dr_1; \quad V \in A_5.$$

Rotation of the rows defines rotation of the columns.

1. V^2 - rotation of the rows to π .
2. $a_{i,j} \rightarrow a_{10-i,10-j}$ or $V^2 = (dr_1)^2$, $V^2 \in A_8$.
3. V^3 - rotation of the rows to $3/2\pi$ clockwise.
4. $a_{i,j} \rightarrow a_{10-j,i}$ or $V^3 = r, d$, $V^3 \in A_4$, $V^4 = E$, $V^4 \in A_1$.
5. Rotation of the columns clockwise: - W $W = r, d$ $W = V^3$, $W^2 = V^2$, $W^3 = V$, $W^4 = E$.
6. F - is a central symmetry with respect to the central element a_{55} .
7. $a_{ij} \rightarrow a_{10-i,10-j}$ or $F = (dr_1)^2$, $F \in A_8$.

It's now easy to see that $F = V^2$. Hence the group S exhausts all possible symmetries of the Sudoku grid.

There are more difficult transformations that can be defined on the Sudoku grid. These transformations can not be reduced to the symmetric ones but being applied they lead to the different Sudoku grid. These more complicated transformations are related to the questions of the unique and non-unique solvability of the particular Sudoku grid and lots of questions are still open in this area of research.